Advanced Mathematical Models & Applications
Vol.8, Special Issue, 2023, pp.401-414

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A CLASS OF DOUBLE PHASE VARIABLE EXPONENT PROBLEMS WITH NONLINEAR BOUNDARY CONDITION

M. El Ahmadi[1*], A. Ayoujil[2], M. Berrajaa[1]

[1]Mohammed I University, FSO, Dept. of Mathematics, LaMAO Laboratory, Oujda, Morocco
[2]Regional Centre of Trades Education and Training, Oujda, Morocco

Abstract. We are interested in the existence and multiplicity of nontrivial weak solutions for a class of double phase problem with variable exponents, where the nonlinearity is superlinear but does not satisfy the $(\mathbf{AR})-$condition. The proofs rely on variational arguments based on the Mountain Pass Theorem and the Fountain Theorem with Cerami condition.

Keywords: Double phase type problems, variational methods, Musielak-Orlicz space, Mountain Pass Theorem, Fountain Theorem, Cerami condition.

AMS Subject Classification: 35J15, 35J62, 35J92. 35P30.

Corresponding author: Mahmoud El Ahmadi, Mohammed I University, FSO, Department of Mathematics, LaMAO Laboratory, Oujda, Morocco, e-mail: *elahmadi.mahmoud@ump.ac.ma*

Received: 31 October 2021; Revised: 30 January 2022; Accepted: 05 March 2022;
Published: 30 November 2023.

1 Introduction

The study of differential equations and variational problems with double phase operator is a new and important topic, since it sheds light on multiple range of applications in the field of mathematical physics such as elasticity theory, strongly anisotropic materials, Lavrentiev's phenomenon, etc. (see Zhikov (1986, 1995); Zhikov et al. (1994); Berdawood et al. (2020)).

In the present paper, we study the existence and multiplicity of solutions for the double phase problem with variable exponents of the following form:

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + a(x)|\nabla u|^{q(x)-2}\nabla u\right) = |u|^{p(x)-2}u + a(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ \left(|\nabla u|^{p(x)-2}u + a(x)|\nabla u|^{q(x)-2}u\right)\cdot \nu = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $1 < p(x) < q(x) < N$ and $\frac{p(x)}{q(x)} < 1 + \frac{1}{N}$, $a : \overline{\Omega} \mapsto [0, +\infty)$ is Lipschitz continuous, ν denotes the outer unit normal of Ω at the point $x \in \partial\Omega$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition.

The differential operator $\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + a(x)|\nabla u|^{q(x)-2}\nabla u\right)$ is called the double phase operator which is a natural generalization of the classical double phase operator when p and q are constant functions $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right)$.

Multiple authors have concentrate on the study of double phase problems recently, and a plethora of results have been obtained Nachaoui et al. (2021); Rasheed et al. (2021). Let us recall some previous results that led us to the present paper.

In Liu & Dai (2018), the authors considered the following problem in the particular case of $p(x) = p$ and $q(x) = q$

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and they proved the existence and multiplicity of nontrivial solutions when the nonlinear term g satisfying the $(\mathbf{AR})-$ condition: i.e.,
$(\mathbf{AR})$ there exist $M > 0$, $\theta > q$ such that for $|t| \geq M$ and a.e. $x \in \Omega$,

$$0 < \theta G(x,t) \leqslant t g(x,t).$$

In Yang et al. (2020), J. Yang, H. Chen and S. Liu have studied the following Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + a(x)|\nabla u|^{q(x)-2}\nabla u\right) = \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Via a variational approach, the existence and multiplicity of solutions have been established, where the nonlinear term g does not satisfy the $(\mathbf{AR})-$ condition.

In the case when $a \equiv 0$, problem (1) becomes a $p(x)$-Laplacian Steklov problem of the form

$$\begin{cases} \Delta_{p(x)} u = |u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial v} = g(x,u) & \text{on } \partial\Omega. \end{cases} \tag{2}$$

By using critical point theory, existence and multiplicity results of problem (2) are proved by A. Ayoujil in Ayoujil (2014).

In a recent paper Cui & Sun (2021), in the case when $p(x) = p$ and $q(x) = q$, Na Cui and Hong-Rui Sun proved that the following problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = f(x,u) - |u|^{p-2}u - a(x)|u|^{q-2}u & \text{in } \Omega, \\ \left(|\nabla u|^{p-2}u + a(x)|\nabla u|^{q-2}u\right)\cdot \nu = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

has a nontrivial weak solution or infinitely many weak solutions with f and g are Carathéodory functions satisfying suitable growth conditions, but do not satisfy the $(\mathbf{AR})-$ condition. Their approach was based on the critical point theory, namely, Mountain Pass Theorem, Fountain Theorem and Clark's Theorem.

Motivated by the above fact, we intend to establish the existence and multiplicity of nontrivial solutions of problem (1), which has never been tackled before. The main novelty, as well as the main difficulty of problem (1) comes from the fact that: on the one hand, the preblem (1) is modeled in the working space $W^{1,\mathcal{H}}(\Omega)$, not just the classical Sobolev space, which needs more delicate and complicated estimates when we consider a nonlinear boundary condition. On the other hand, the exponents p and q are nonconstant functions, then, problem (1) has a more complicated structure.

Before stating our main results, we need to make the following assumptions of g:
(H_1) There exists $C > 0$ such that

$$|g(x,t)| \leq C(1 + |t|^{\alpha(x)-1}) \quad \text{for all } (x,t) \in \partial\Omega \times \mathbb{R},$$

where $\alpha \in C_+(\partial\Omega)$, $1 < q^+ < \alpha^- \leq \alpha(x) \leq \alpha^+ < p_*(x)$ and

$$p_*(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

(H_2) $\liminf\limits_{|t|\to\infty} \frac{g(x,t)t}{|t|^{q^+}} = +\infty$ uniformly for a.e. $x \in \partial\Omega$,
where $G(x,t) = \int_0^t g(x,s)ds$.
(H_3) $\lim\limits_{t\to 0} \frac{g(x,t)}{|t|^{q^+-1}} = 0$ uniformly for a.e. $x \in \partial\Omega$.
(H_4) There exists a constant $\mu \geq 1$, such that for all $(s,t) \in [0,1] \times \mathbb{R}$, for each $\mathcal{F}_\lambda \in \mathcal{B}$ and for all $\theta \in [p^-, q^+]$, the inequality

$$\mu \mathcal{F}_\lambda(x,t) \geq \mathcal{F}_\theta(x,st) \quad \text{holds for a.e. } x \in \partial\Omega,$$

where $\mathcal{B}$ is the class of functions defined by $\mathcal{B} = \Big\{ \mathcal{F}_\lambda \mid \mathcal{F}_\lambda = g(x,t)t - \lambda G(x,t), \lambda \in [p^-, q^+] \Big\}$.
(H_5) $g(x,-t) = -g(x,t)$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$.

It is known that the main role of the famous Ambrosetti-Rabinowitz type condition is to ensure the boundendess of the Palais-Smale type sequences of the corresponding functional. However, there are several functions which are superlinear at infinity and at the origin but do not satisfy the **(AR)**-condition. For example, the function

$$g(x,t) = q|t|^{q-2}t\ln(1+|t|^2)$$

does not satisfy the (**AR**)-condition, but it satisfies $(H_1) - (H_5)$.

Remark 1. 1. *The hypothesis (H_4), which is important in obtaining a compactness condition of Palais–Smale type, can be found in Zang (2008).*

2. *In general, the (**AR**)$-$ condition does not imply (H_4). Indeed, if $p(x) = p$ and $q(x) = q$, then, hypothesis (H_4) becomes:*
(H_4) There exists a constant $\mu \geq 1$, such that for all $(s,t) \in [0,1] \times \mathbb{R}$, for each $\mathcal{F}_\lambda \in \mathcal{B}$ and for all $\theta \in [p,q]$, the inequality

$$\mu \mathcal{F}_\lambda(x,t) \geq \mathcal{F}_\theta(x,st) \quad \textit{holds for a.e. } x \in \partial\Omega,$$

where $\mathcal{B} = \Big\{ \mathcal{F}_\lambda \mid \mathcal{F}_\lambda = g(x,t)t - \lambda G(x,t), \lambda \in [p,q] \Big\}$.
Let us consider the following assumption:
$(H_4^{'})$ There exists a constant $\mu \geq 1$, such that for all $(s,t) \in [0,1] \times \mathbb{R}$ the inequality

$$\mu \mathcal{F}_q(x,t) \geq \mathcal{F}_q(x,st) \quad \textit{holds for a.e. } x \in \partial\Omega.$$

*From Zang (2008), the function $g(x,t) = (q+2)|t|^q t + (q+1)|t|^{q-1} t \sin^2 \frac{1}{t} - |t|^{q-1} \sin \frac{1}{t} \cos \frac{1}{t}$ satisfies the (**AR**)$-$ condition, but does not satisfy $(H_4^{'})$. Since (H_4) implies $(H_4^{'})$, by contraposition, we conclude that g does not satisfy (H_4).*

Now we are ready to state our main results.

Theorem 1. *Assume $(H_1), (H_2)$, (H_3) and (H_4). Then problem* (1) *has at least one nontrivial solution in $W^{1,\mathcal{H}}(\Omega)$.*

Theorem 2. *Assume $(H_1), (H_2)$, (H_4) and (H_5). Then problem* (1) *possesses a sequence of nontrivial weak solutions (u_n) in $W^{1,\mathcal{H}}(\Omega)$ such that $J(u_n) \to +\infty$ as $n \to +\infty$.*

2 Preliminaries

To study double phase problems, we need some definitions and basic properties of $W^{1,\mathcal{H}}(\Omega)$ which are called Musielak–Orlicz–Sobolev space. For more details, see Benkirane & Sidi El Vally (2014); Colasuonno & Squassina (2016); Fan (2012); Harjulehto & Hasto (2019); Musielak (1983) and references therein.

Denote by $N(\Omega)$ the set of all generalized N-functions (N stands for nice). Let us denote by

$$\mathcal{H} : \Omega \times [0,+\infty[\to [0,+\infty[$$

the functional defined as

$$\mathcal{H}(x,t) = t^{p(x)} + a(x)t^{q(x)}, \quad \text{for all } \ (x,t) \in \Omega \times [0,+\infty[\,,$$

where the weight function $a(.)$ and the variable exponents $p(x)$, $q(x)$ satisfies the following hypothesis:

$$H(a): \ p,q \in C_+(\overline{\Omega}) \text{ such that } p(x) < q(x) < N \text{ for all } x \in \overline{\Omega} \text{ and } 0 \leq a(.) \in L^1(\Omega). \tag{3}$$

It is clear that $\mathcal{H}$ is a generalized N-function, locally integrable and

$$\mathcal{H}(x,2t) \leq 2^{q^+}\mathcal{H}(x,t), \quad \text{for all } \ (x,t) \in \Omega \times [0,+\infty[\,,$$

which is called condition (Δ_2).

we designate the Musielak–Orlicz space by

$$L^{\mathcal{H}}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \mathcal{H}(x,|u|)dx < +\infty\Big\},$$

equipped with the so-called Luxemburg norm

$$|u|_{\mathcal{H}} = \inf\Big\{\lambda > 0 : \int_\Omega \mathcal{H}(x,|\frac{u}{\lambda}|)dx \leq 1\Big\}.$$

The Musielak–Orlicz–Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined as

$$W^{1,\mathcal{H}}(\Omega) = \Big\{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\Big\},$$

endowed with the norm

$$\| u \|_{1,\mathcal{H}} = |u|_{\mathcal{H}} + |\nabla u|_{\mathcal{H}}.$$

With such norms, $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$ are separable, uniformly convex, and reflexive Banach space.

On $L^{\mathcal{H}}(\Omega)$, we consider the function $\rho_{\mathcal{H}} : L^{\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{\mathcal{H}}(u) = \int_\Omega \Big(|u|^{p(x)} + a(x)|u|^{q(x)}\Big)\,dx.$$

The relationship between $\rho_{\mathcal{H}}$ and $|.|_{\mathcal{H}}$ is established by the next result.

Proposition 1. *(See Crespo-Blanco et al. (2021)) For $u \in L^{\mathcal{H}}(\Omega)$, $(u_n) \subset L^{\mathcal{H}}(\Omega)$ and $\lambda > 0$, we have*

1. *For $u \neq 0$, $|u|_{\mathcal{H}} = \lambda \Longleftrightarrow \rho_{\mathcal{H}}(\frac{u}{\lambda}) = 1$;*
2. *$|u|_{\mathcal{H}} < 1\,(= 1, > 1) \Longleftrightarrow \rho_{\mathcal{H}}(u) < 1\,(= 1, > 1)$;*
3. *$|u|_{\mathcal{H}} > 1 \Longrightarrow |u|_{\mathcal{H}}^{p^-} \leq \rho_{\mathcal{H}}(u) \leq |u|_{\mathcal{H}}^{q^+}$;*

4. $|u|_{\mathcal{H}} < 1 \Longrightarrow |u|_{\mathcal{H}}^{q^+} \leq \rho_{\mathcal{H}}(u) \leq |u|_{\mathcal{H}}^{p^-}$;
5. $\lim\limits_{n\to+\infty} |u_n|_{\mathcal{H}} = 0 \Leftrightarrow \lim\limits_{n\to+\infty} \rho_{\mathcal{H}}(u_n) = 0$ *and* $\lim\limits_{n\to+\infty} |u_n|_{\mathcal{H}} = +\infty \Leftrightarrow \lim\limits_{n\to+\infty} \rho_{\mathcal{H}}(u_n) = +\infty$.

On $W^{1,\mathcal{H}}(\Omega)$, we introduce the equivalent norm by

$$\|u\| := \inf\left\{\lambda > 0 : \int_\Omega \left[\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + a(x)\left|\frac{\nabla u}{\lambda}\right|^{q(x)} + \left|\frac{u}{\lambda}\right|^{p(x)} + a(x)\left|\frac{u}{\lambda}\right|^{q(x)}\right] \mathrm{d}x \leq 1\right\}$$

Similar to the Proposition (1), we have

Proposition 2. *(See Crespo-Blanco et al. (2021)) Let*

$$\hat{\rho}_{\mathcal{H}}(u) = \int_\Omega \left(|\nabla u|^{p(x)} + a(x)|\nabla u|^{q(x)}\right)dx + \int_\Omega \left(|u|^{p(x)} + a(x)|u|^{q(x)}\right) dx$$

. *For* $u \in W^{1,\mathcal{H}}(\Omega)$, $(u_n) \subset W^{1,\mathcal{H}}(\Omega)$ *and* $\lambda > 0$, *we have*

1. *For* $u \neq 0$, $\|u\| = \lambda \Longleftrightarrow \hat{\rho}_{\mathcal{H}}(\frac{u}{\lambda}) = 1$;
2. $\|u\| < 1\,(=1, >1) \Longleftrightarrow \hat{\rho}_{\mathcal{H}}(u) < 1\,(=1, >1)$;
3. $\|u\| > 1 \Longrightarrow \|u\|^{p^-} \leq \hat{\rho}_{\mathcal{H}}(u) \leq \|u\|^{q^+}$;
4. $\|u\| < 1 \Longrightarrow \|u\|^{q^+} \leq \hat{\rho}_{\mathcal{H}}(u) \leq \|u\|^{p^-}$;
5. $\lim\limits_{n\to+\infty} \|u_n\| = 0 \Leftrightarrow \lim\limits_{n\to+\infty} \hat{\rho}_{\mathcal{H}}(u_n) = 0$ *and* $\lim\limits_{n\to+\infty} \|u_n\|_{\mathcal{H}} = +\infty \Leftrightarrow \lim\limits_{n\to+\infty} \hat{\rho}_{\mathcal{H}}(u_n) = +\infty$.

We define the weighted space

$$L^{q(x)}_{\mu(x)}(\partial\Omega) = \left\{u : \partial\Omega \to \mathbb{R} \text{ is measurable and } \int_{\partial\Omega} \mu(x)|u|^{q(x)}\, d\sigma < +\infty\right\}$$

with the seminorm

$$|u|_{q(x),\mu(x)} = \inf\left\{\lambda > 0 : \int_{\partial\Omega} \mu(x)\left(\frac{|u|}{\lambda}\right)^{q(x)} d\sigma \leq 1\right\}.$$

In particular, when $\mu \equiv 1$ on $\partial\Omega$, the space $L^{q(x)}_{\mu(x)}(\partial\Omega)$ becomes a variable exponent Lebesgue space $L^{q(x)}(\partial\Omega)$ with $|u|_{q(x),\mu(x)} = |u|_{q(x),\partial\Omega}$.

Recall the following embedding results.

Proposition 3. *(See Crespo-Blanco et al. (2021)) Let hypothesis* (3) *be satisfied. Then the following embeddings hold:*

1. *If* $p \in C_+(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ *for some* $\gamma \geq N$. *Then, there is a continuous embedding* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ *for* $r \in C(\partial\Omega)$ *with* $1 \leq r(x) \leq p_*(x)$ *for all* $x \in \partial\Omega$.
2. *There is a compact embedding* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ *for* $r \in C(\partial\Omega)$ *with* $1 \leq r(x) < p_*(x)$ *for all* $x \in \partial\Omega$.

Let $A : W^{1,\mathcal{H}}(\Omega) \to \left(W^{1,\mathcal{H}}(\Omega)\right)^*$ be defined by

$$\langle A(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + a(x)|\nabla u|^{q(x)-2}\right) \nabla u.\nabla v\; dx + \int_\Omega \left(|u|^{p(x)-2} + a(x)|u|^{q(x)-2}\right) u.v\; dx$$

for all $u, v \in W^{1,\mathcal{H}}(\Omega)$, where $\left(W^{1,\mathcal{H}}(\Omega)\right)^*$ denotes the dual space of $W^{1,\mathcal{H}}(\Omega)$ and $\langle .,. \rangle$ stands for the duality pairing between $W^{1,\mathcal{H}}(\Omega)$ and $\left(W^{1,\mathcal{H}}(\Omega)\right)^*$.

Proposition 4. *(See Crespo-Blanco et al. (2021)) Let hypothesis* (3) *be satisfied.*

1. *The operator* $A : W^{1,\mathcal{H}}(\Omega) \to \left(W^{1,\mathcal{H}}(\Omega)\right)^*$ *is continuous, bounded and strictly monotone.*
2. *The operator* $A : W^{1,\mathcal{H}}(\Omega) \to \left(W^{1,\mathcal{H}}(\Omega)\right)^*$ *satisfies the* (S_+)*-property, i.e., if* $u_n \rightharpoonup u$ *in* $W^{1,\mathcal{H}}(\Omega)$ *and* $\overline{\lim_{n\to+\infty}} \langle J(u_n) - J(u), u_n - u\rangle \leq 0$, *then* $u_n \to u$ *in* $W^{1,\mathcal{H}}(\Omega)$.
3. *The operator* $A : W^{1,\mathcal{H}}(\Omega) \to \left(W^{1,\mathcal{H}}(\Omega)\right)^*$ *is coercive and a homeomorphism.*

From now on, we denote by $E = W^{1,\mathcal{H}}(\Omega)$ and $E^* = \left(W^{1,\mathcal{H}}(\Omega)\right)^*$ the dual space.
The problem (1) has a variational structure, its associated energy functional $J : E \to \mathbb{R}$ is defined as follows

$$J(u) = I(u) - \varphi(u),$$

where

$$I(u) = \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{a(x)}{q(x)}|\nabla u|^{q(x)}\right) \, dx + \int_\Omega \left(\frac{1}{p(x)}|u|^{p(x)} + \frac{a(x)}{q(x)}|u|^{q(x)}\right) \, dx,$$

and

$$\varphi(u) = \int_{\partial\Omega} G(x,u) \, d\sigma,$$

where $d\sigma$ is the measure on the boundary.
Then, it follows from the hypothesis (H_1) that the functional $J \in C^1(E,\mathbb{R})$, and its Fréchet derivative is

$$\langle J'(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + a(x)|\nabla u|^{q(x)-2}\right) \nabla u.\nabla v \, dx + \int_\Omega \left(|u|^{p(x)-2} + a(x)|u|^{q(x)-2}\right) u.v \, dx -$$

$$- \int_{\partial\Omega} g(x,u)v \, d\sigma,$$

for any $u, v \in E$.

Definition 1. *Let* $u \in E$. *We say that* u *is a* ***weak solution*** *of the problem* (1) *if*

$$\int_\Omega \left(|\nabla u|^{p(x)-2} + a(x)|\nabla u|^{q(x)-2}\right) \nabla u.\nabla v \, dx + \int_\Omega \left(|u|^{p(x)-2} + a(x)|u|^{q(x)-2}\right) u.v \, dx -$$

$$- \int_{\partial\Omega} g(x,u)v \, d\sigma = 0,$$

for all $v \in E$.

Definition 2. *(See Cerami (1978)) Let* $(X, \|.\|)$ *be a real Banach space and* $\phi \in C^1(X,\mathbb{R})$. *Given* $c \in \mathbb{R}$, *we say that* ϕ *satisfies the Cerami condition (we denote* $(C_c)-$ *condition) if*
a) any bounded sequence $(u_n) \subset X$ *such that* $\phi(u_n) \to c$ *and* $\phi'(u_n) \to 0$ *has a convergent subsequence;*
b) there exist constants $\alpha, \beta, r > 0$ *such that*

$$\|\phi'(u)\| \, \|u\| \geq \beta, \quad \forall u \in \phi^{-1}\left([c-\alpha, c+\alpha]\right) \quad \text{with} \quad \|u\| \geq r.$$

If this condition is satisfied at every level $c \in \mathbb{R}$, *then, we say that* J *satisfies* (C)*-condition.*

Now, we present the following theorems which will play a fundamental role in the proof of main Theorems.

Theorem 3. *(See Bartolo et al. (1983)) Let X be a real Banach space, let $\phi : X \to \mathbb{R}$ be a functional of class $C^1(X, \mathbb{R})$ that satisfies (C)-condition, $\phi(0) = 0$ and the following conditions hold:*

1. *There exist positive constant ρ and α such that $\phi(u) \geq \alpha$ for any $u \in X$ with $\|u\| = \rho$.*
2. *There exists a function $e \in X$ such that $\|e\| > \rho$ and $\phi(e) \leq 0$.*

Then, the functional ϕ has a critical value $c \geq \alpha$, that is, there exists $u \in X$ such that $\phi(u) = c$ and $\phi'(u) = 0$ in X^*.

Let X be a real, reflexive, and Banach space, it is known Zhao (1991) that for a separable and reflexive Banach space there exist $\{e_j\}_{j\in\mathbb{N}} \subset X$ and $\{e_j^*\}_{j\in\mathbb{N}} \subset X^*$ such that

$$X = \overline{span\{e_j : j = 1, 2, ...\}}, \quad X^* = \overline{span\{e_j^* : j = 1, 2, ...\}},$$

and $\langle e_i^*, e_j\rangle = 1$ if $i = j$, $\langle e_i^*, e_j\rangle = 0$ if $i \neq j$.
We denote $X_j = span\{e_j\}$,$Y_k = \bigoplus_{j=1}^{k} X_j$ and $Z_k = \overline{\bigoplus_{j=k}^{+\infty} X_j}$.

Theorem 4. *(See Zou (2001)) Assume that X is a Banach space, and let $\phi : X \to \mathbb{R}$ be an even functional of class $C^1(X, \mathbb{R})$ and satisfies $(C)-$ condition . For every $k \in \mathbb{N}$, there exists $\gamma_k > \eta_k > 0$ such that*
(A_1) $b_k := \inf\{\phi(u) : u \in Z_k, \|u\| = \eta_k\} \to +\infty$ as $k \to +\infty$;
(A_2) $c_k := \max\{\phi(u) : u \in Y_k, \|u\| = \gamma_k\} \leq 0$.
Then, ϕ has a sequence of critical values tending to $+\infty$.

3 Compactness condition for the energy functional corresponding to problem (1)

In this section, we present the following compactness result which will play a crucial role in the proof of main Theorems.

Lemma 1. *Assume that $(\mathbf{H_1}), (\mathbf{H_2})$ and $(\mathbf{H_4})$ hold. Then, J satisfies the (C)-condition.*

Proof. Firstly, we show that J satisfies the first assertion of $(C)-$condition. Let $(u_n) \subset E$ be a bounded sequence such that

$$J(u_n) \to c, \ c \in \mathbb{R} \quad \text{and} \quad J'(u_n) \to 0.$$

As E is reflexive, for a subsequence still denoted by (u_n), we have $u_n \rightharpoonup u$ in E. By (H_1), using arguments analogous to those in Fan & Zhang (2003), it can be seen that the functional $\varphi : E \to E^*$ is completely continuous, then $\varphi(u_n) \to \varphi(u)$. Since $J'(u_n) = I'(u_n) - \varphi'(u_n) \to 0$, we get that $I'(u_n) \to \varphi'(u)$. Using the fact that $I' \equiv A$ is a homeomorphism in view of Proposition (4), then we obtain that $u_n \to u$ in E.

Now we check that J satisfies the second assertion of (C)-condition. To this end, arguing by contradiction, it is assumed that there exist $c \in \mathbb{R}$ and $(u_n) \subset E$ satisfying

$$J(u_n) \to c, \quad \|u_n\| \to +\infty \quad \text{and} \quad \|J'(u_n)\| \, \|u_n\| \to 0. \tag{4}$$

We can suppose that $\|u_n\| > 1$, for $n \in \mathbb{N}$, then we obtain

$$\begin{aligned} c &= \lim_{n\to+\infty} \left[J(u_n) - \frac{1}{\delta_n} \langle J'(u_n), u_n\rangle \right] \\ &= \lim_{n\to+\infty} \left[\frac{1}{\delta_n} \int_{\partial\Omega} g(x, u_n) u_n d\sigma - \int_{\partial\Omega} G(x, u_n) d\sigma \right], \end{aligned} \tag{5}$$

where $\delta_n = \frac{\hat{\rho}_{\mathcal{H}}(u_n)}{I(u_n)}$.
Let a sequence (υ_n) be defined by $\upsilon_n = \frac{u_n}{\|u_n\|}$, then, $\|\upsilon_n\| = 1$. Since E is reflexive and $(\upsilon_n) \subset E$ is a bounded sequence, we assume that for some subsequence, still denoted by itself, there is $\upsilon \in E$ such that

$$\begin{aligned} \upsilon_n &\rightharpoonup \upsilon \quad \text{in } E, \\ \upsilon_n &\to \upsilon \quad \text{in } L^{\alpha(x)}(\partial\Omega), \\ \upsilon_n &\to \upsilon \quad \text{in } L^{q^+}(\partial\Omega), \\ \upsilon(x) &\to \upsilon(x) \quad \text{a.e. } x \in \partial\Omega, \end{aligned} \tag{6}$$

where α comes from (H_1).
Next, we need to distinguish two cases: $\upsilon = 0$ and $\upsilon \neq 0$.
Case1. If $\upsilon = 0$, according to the proof of Lemma 3.6 in Jeanjean (1999), we can define a sequence $(t_n) \subset \mathbb{R}$ such that

$$J(t_n u_n) = \max_{t\in[0,1]} J(tu_n). \tag{7}$$

Fix $L > 0$ such that $L > \frac{1}{2p^+}$. Let $\overline{\upsilon_n} = (2q^+L)^{\frac{1}{p^-}}\upsilon_n$. By (6), it is seen that

$$\overline{\upsilon_n} \to 0 \quad \text{in} \quad L^{\alpha(x)}(\partial\Omega).$$

Using (H_1), it follows that

$$|G(x,t)| \leq C(|t| + |t|^{\alpha(x)}).$$

As the function $t \mapsto G(.,t)$ is continuous, we obtain

$$G(.,\overline{\upsilon_n}) \to 0, \quad \text{as} \quad n \to +\infty \quad \text{in} \quad L^1(\partial\Omega).$$

Therefore,

$$\lim_{n\to+\infty} \int_{\partial\Omega} G(x,\overline{\upsilon_n})d\sigma = 0. \tag{8}$$

Because $\|u_n\| \to +\infty$, it is obvious that $\frac{(2p^+L)^{\frac{1}{q^-}}}{\|u_n\|} \in]0,1[$. As $p^- \leq p(x) < q(x) \leq q^+$, then, for n large enough, we obtain

$$\begin{aligned} J(t_n u_n) &\geq J(\overline{\upsilon_n}) \\ &= \int_\Omega \left(\frac{1}{p(x)}|\nabla\overline{\upsilon_n}|^{p(x)} + \frac{a(x)}{q(x)}|\nabla\overline{\upsilon_n}|^{q(x)}\right) dx + \int_\Omega \left(\frac{1}{p(x)}|\overline{\upsilon_n}|^{p(x)} + \frac{a(x)}{q(x)}|\overline{\upsilon_n}|^{q(x)}\right) dx - \int_{\partial\Omega} G(x,\overline{\upsilon_n})\, d\sigma \\ &\geq \frac{2q^+L}{q^+}\int_\Omega \left(|\nabla\upsilon_n|^{p(x)} + a(x)|\nabla\upsilon_n|^{q(x)}\right) dx + \frac{2q^+L}{q^+}\int_\Omega \left(|\upsilon_n|^{p(x)} + a(x)|\upsilon_n|^{q(x)}\right) dx - \int_{\partial\Omega} G(x,\overline{\upsilon_n})\, d\sigma \\ &= 2L\hat{\rho}_{\mathcal{H}}(\upsilon_n) - \int_{\partial\Omega} G(x,\overline{\upsilon_n})\, d\sigma \\ &\geq 2L - \int_{\partial\Omega} G(x,\overline{\upsilon_n})\, d\sigma \geq 2L, \end{aligned} \tag{9}$$

that is,

$$\lim_{n\to+\infty} J(t_n u_n) = +\infty. \tag{10}$$

Since $J(0) = 0$ and $\lim\limits_{n\to+\infty} J(u_n) = c$, then, $t_n \in]0,1[$ and

$$\hat{\rho}_{\mathcal{H}}(t_n u_n) - \int_{\partial\Omega} g(x,t_n u_n)t_n u_n\, d\sigma = \langle J'(t_n u_n), t_n u_n\rangle = t_n \frac{d}{dt}\bigg|_{t=t_n} J(tu_n) = 0, \tag{11}$$

when n is large enough.
Therefore, combining (10) and (11), we get

$$\begin{aligned}\int_{\partial\Omega}\left(\frac{1}{\delta_{t_n}}g(x,t_nu_n)t_nu_n-G(x,t_nu_n)\right)\,d\sigma &= \frac{1}{\delta_{t_n}}\hat{\rho}_{\mathcal{H}}(t_nu_n)-\int_{\partial\Omega}G(x,t_nu_n)\,d\sigma\\ &= J(t_nu_n)\underset{n\to+\infty}{\longrightarrow}+\infty,\end{aligned}\tag{12}$$

with $\delta_{t_n}=\frac{\hat{\rho}_{\mathcal{H}}(t_nu_n)}{I(t_nu_n)}$.

By simple calculation, we have $\delta_n,\delta_{t_n}\in[p^-,q^+]$. Then, $\mathcal{F}_{\delta_n},\mathcal{F}_{\delta_{t_n}}\in\mathcal{B}$. Accordingly, from (H_4), we have

$$\begin{aligned}\int_{\partial\Omega}\left(\frac{1}{\delta_n}g(x,u_n)u_n-G(x,u_n)\right)\,d\sigma &= \frac{1}{\delta_n}\int_{\partial\Omega}\mathcal{F}_{\delta_n}(x,u_n)\,d\sigma\\ &\geq \frac{1}{\mu\delta_n}\int_{\partial\Omega}\mathcal{F}_{\delta_{t_n}}(x,t_nu_n)\,d\sigma\\ &= \frac{\delta_{t_n}}{\mu\delta_n}\int_{\partial\Omega}\left(\frac{1}{\delta_{t_n}}g(x,t_nu_n)t_nu_n-G(x,t_nu_n)\right)\,d\sigma.\end{aligned}$$

Since $\inf_n\frac{\delta_{t_n}}{\mu\delta_n}>0$, by (12), we can deduce that

$$\int_{\partial\Omega}\left(\frac{1}{\delta_n}g(x,u_n)u_n-G(x,u_n)\right)\,d\sigma\to+\infty,\ \text{as}\ \ n\to+\infty,$$

which is contradiction to (5).
Case2. If $\upsilon\neq0$, by (4) and Proposition (2), we write

$$\hat{\rho}_{\mathcal{H}}(u_n)-\int_{\partial\Omega}g(x,u_n)u_n\,d\sigma=\langle J'(u_n),u_n\rangle=\circ(1)\|u_n\|,\tag{13}$$

then

$$\begin{aligned}1-\circ(1) &= \int_{\partial\Omega}\frac{g(x,u_n)u_n}{\hat{\rho}_{\mathcal{H}}(u_n)}\,d\sigma\\ &\geq \int_{\partial\Omega}\frac{g(x,u_n)u_n}{\|u_n\|^{q^+}}\,d\sigma\\ &= \int_{\partial\Omega}\frac{g(x,u_n)u_n}{|u_n|^{q^+}}|\upsilon_n|^{q^+}\,d\sigma.\end{aligned}\tag{14}$$

Let's define the set $V_0=\{x\in\partial\Omega:\upsilon(x)\neq0\}$. If $x\in V_0$, then

$$\lim_{n\to+\infty}\upsilon_n(x)=\lim_{n\to+\infty}\frac{u_n(x)}{\|u_n\|}=\upsilon(x)\neq0.$$

Therefore, $|u_n(x)|=|\upsilon_n(x)|\;\|u_n\|\underset{n\to+\infty}{\longrightarrow}+\infty\quad a.e.\ x\in\partial\Omega$.
Hence, using (H_2), we see

$$\frac{g(x,u_n)u_n}{|u_n|^{q^+}}|\upsilon_n|^{q^+}\underset{n\to+\infty}{\longrightarrow}+\infty.$$

As $|V_0|>0$, by Fatou's lemma, we conclude

$$\int_{V_0}\frac{g(x,u_n)u_n}{|u_n|^{q^+}}|\upsilon_n|^{q^+}\,d\sigma\underset{n\to+\infty}{\longrightarrow}+\infty.\tag{15}$$

From (H_1) and (H_2), it follows that there exists a real number D such that $\frac{g(x,t)t}{t^{q^+}} \geq D$ for any $t \in \mathbb{R}$ and a.e. $x \in \partial\Omega$. Moreover, we have

$$\int_{\partial\Omega\setminus V_0} |\upsilon_n|^{q^+} \, d\sigma \underset{n\to+\infty}{\longrightarrow} 0.$$

Then, there exists $N > -\infty$ such that

$$\int_{\partial\Omega\setminus V_0} \frac{g(x,u_n)u_n}{|u_n|^{q^+}} |\upsilon_n|^{q^+} \, d\sigma \geq D \int_{\partial\Omega\setminus V_0} |\upsilon_n|^{q^+} \, d\sigma \geq N > -\infty. \tag{16}$$

Combining (15) and (16) gives

$$\int_{\partial\Omega} \frac{g(x,u_n)u_n}{|u_n|^{q^+}} |\upsilon_n|^{q^+} \, d\sigma = \int_{V_0} \frac{g(x,u_n)u_n}{|u_n|^{q^+}} |\upsilon_n|^{q^+} \, d\sigma + \int_{\partial\Omega\setminus V_0} \frac{g(x,u_n)u_n}{|u_n|^{q^+}} |\upsilon_n|^{q^+} \, d\sigma \to +\infty, \text{ as } n \to +\infty.$$

This leads to a contradiction with (14). Consequently, the functional J satisfies the second assertion of $(C)-$ condition. The proof is complete. □

4 Proofs of main results

The main aim of this section is to prove our main results.

4.1 Proof of Theorem (1)

The proof is based on the Mountain Pass Theorem (3). Let $X = E$ and $\phi \equiv J$. Obviously, by Lemma (1), J satisfies the $(C)-$condition. Firstly, we will show that J possesses the mountain pass geometry.

Lemma 2. *There exist $\eta, \rho > 0$ such that*

$$I(u) \geq \eta, \quad \text{for } \|u\| = \rho. \tag{17}$$

Proof. Let $\varepsilon > 0$. By assumptions (H_1) and (H_3), it follows that

$$|G(x,t)| \leq \varepsilon |t|^{q^+} + C_\varepsilon |t|^{\alpha(x)}, \quad \text{for all } (x,t) \in \partial\Omega \times \mathbb{R}.$$

Then, for $\|u\|$ sufficiently small, we have

$$\begin{aligned} J(u) &= \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u|^{q(x)} \right) dx + \int_\Omega \left(\frac{1}{p(x)} |u|^{p(x)} + \frac{a(x)}{q(x)} |u|^{q(x)} \right) dx - \int_{\partial\Omega} G(x,u) \, d\sigma \\ &\geq \frac{1}{q^+} \|u\|^{q^+} - \varepsilon \int_{\partial\Omega} |u|^{q^+} \, d\sigma - C_\varepsilon \int_{\partial\Omega} |u|^{\alpha(x)} \, d\sigma. \end{aligned}$$

Since $1 < q^+ < \alpha^- \leq \alpha(x) \leq \alpha^+ < p_*(x)$ for all $x \in \partial\Omega$, according to the Proposition (3), we have

$$E \hookrightarrow L^{q^+}(\partial\Omega) \quad \text{and} \quad E \hookrightarrow L^{\alpha(x)}(\partial\Omega),$$

with a compact embeddings. Thus, there exist $c_1, c_2 > 0$ such that

$$|u|_{q^+,\partial\Omega} \leq c_1 \|u\| \quad \text{and} \quad |u|_{\alpha(x),\partial\Omega} \leq c_2 \|u\|, \quad \text{for all } u \in E. \tag{18}$$

Consequently, we have

$$J(u) \geq \frac{1}{q^+} \|u\|^{q^+} - \varepsilon c_1^{q^+} \|u\|^{q^+} - C_\varepsilon c_2^{\alpha^-} \|u\|^{\alpha^-}. \tag{19}$$

We can choose ε such that $\varepsilon c_1^{q^+} < \frac{1}{2q^+}$. Therefore, as $\alpha^- > q^+$, we obtain

$$J(u) \geq \|u\|^{q^+}\left(\frac{1}{2q^+} - C_\varepsilon c_2^{\alpha^-}\|u\|^{\alpha^- - q^+}\right) = \rho^{q^+}\left(\frac{1}{2q^+} - C_\varepsilon c_2^{\alpha^-}\rho^{\alpha^- - q^+}\right) = \eta > 0.$$

□

Lemma 3. *There exists $e \in E$ with $\|e\| > \rho$ such that $J(e) < 0$.*

Proof. Let $u \in E \setminus \{0\}$. By (H_2), we can choose a constant A and C_A such that

$$A > \frac{\hat{\rho}_{\mathcal{H}}(u)}{p^- \int_{\partial\Omega} |u|^{q^+} dx},$$

so that

$$G(x,t) \geq A|t|^{q^+}, \quad \text{for all } \ |t| > C_A \ \text{ and uniformaly in } \ \partial\Omega. \tag{20}$$

Let $t > 1$ be large enough, we have

$$\begin{aligned} J(tu) &= \int_\Omega \left(\frac{1}{p(x)}|\nabla tu|^{p(x)} + \frac{a(x)}{q(x)}|\nabla tu|^{q(x)}\right) dx + \int_\Omega \left(\frac{1}{p(x)}|tu|^{p(x)} + \frac{a(x)}{q(x)}|tu|^{q(x)}\right) dx - \int_{\partial\Omega} G(x,tu)\, d\sigma \\ &\leq \frac{t^{q^+}}{p^-}\left[\int_\Omega \left(|\nabla u|^{p(x)} + a(x)|\nabla u|^{q(x)}\right) dx + \int_\Omega \left(|u|^{p(x)} + a(x)|u|^{q(x)}\right) dx\right] - \int_{\{|tu| \leq C_A} G(x,tu)\, d\sigma \\ &- \int_{\{|tu| > C_A} G(x,tu)\, d\sigma. \end{aligned}$$

Since $G(x,.)$ is continuous in $t \in [-C_A, C_A]$, there exists a positive constant C_0 such that

$$|G(x,s)| \leq C_0, \quad \text{for all } \ (x,s) \in \partial\Omega \times [-C_A, C_A]\,. \tag{21}$$

Then, using (20) and (21), it follows that

$$\begin{aligned} J(tu) &\leq \frac{t^{q^+}}{p^-}\hat{\rho}_{\mathcal{H}}(u) - At^{q^+}\int_{\partial\Omega} |u|^{q^+}\, d\sigma + C_0|\partial\Omega| \\ &= t^{q^+}\left(\frac{\hat{\rho}_{\mathcal{H}}(u)}{p^-} - A\int_{\partial\Omega} |u|^{q^+}\, d\sigma\right) + C_0|\partial\Omega| \\ &\longrightarrow -\infty, \quad \text{ as } \ t \to +\infty. \end{aligned}$$

Accordingly, there exist $t_1 > 1$ and $e = t_1 u \in E$ such that $\|e\| > \rho$ and $J(e) < 0$. □

Finally, by lemmas (2) - (3), and the fact that $J(0) = 0$, J satisfies the Mountain Pass Theorem. Therefore, the problem (1) has a nontrivial weak solution in $W^{1,\mathcal{H}}(\Omega)$.

4.2 Proof of Theorem (2)

Let $X = E$ and $\phi \equiv J$. Evidently, by (H_5) and Lemma (1), J is an even functional and satisfies the $(C)-$ condition. To apply the Fountain Theorem (4), it suffices to show that there exist $\gamma_k > \eta_k > 0$ such that
(A_1) $b_k := \inf\{J(u) : u \in Z_k, \|u\| = \eta_k\} \to +\infty$ as $k \to +\infty$;
(A_2) $c_k := \max\{J(u) : u \in Y_k, \|u\| = \gamma_k\} \leq 0$.

We first give the following lemmas that will be used later.

Lemma 4. *If $\beta \in C_+(\partial\Omega)$, $\beta(x) < p_*(x)$ for every $x \in \partial\Omega$. Put*

$$d_k = \sup_{\|u\|=1, u\in Z_k} |u|_{\beta(x),\partial\Omega},$$

then $\lim_{k\to+\infty} d_k = 0$.

Proof. Suppose by contradiction that there exist $\varepsilon > 0, k_1 > 0$ and $(u_k) \subset Z_k$ such that

$$\|u_k\| = 1 \quad \text{and} \quad |u_k|_{\beta(x),\partial\Omega} \geq \varepsilon,$$

for every $k \geq k_1$. Since (u_k) is bounded in E, then, there exists $u \in E$ such that

$$u_k \underset{k\to\infty}{\rightharpoonup} u \quad \text{in} \quad E \quad \text{and} \quad \langle e_i^*, u\rangle = \lim_{k\to\infty} \langle e_i^*, u_k\rangle = 0,$$

for $i = 1, 2...$
Thus, $u = 0$. However, we obtain:

$$\varepsilon \leq \lim_{k\to\infty} |u_k|_{\beta(x),\partial\Omega} = |u|_{\beta(x),\partial\Omega} = 0,$$

which is a contradiction. □

Lemma 5. *For all $\varrho \in C_+(\partial\Omega)$ and $u \in L^{\varrho(x)}(\partial\Omega)$, there is $\zeta \in \partial\Omega$ such that*

$$\int_{\partial\Omega} |u|^{\varrho(x)}\, d\sigma = |u|^{\varrho(\zeta)}_{\varrho(x),\partial\Omega}.$$

Verification of $(\mathbf{A_1})$. Let $u \in Z_k$ with $\|u\| = R_k = (c_6\alpha^- d_k^{\alpha^+})^{\frac{1}{p^- - \alpha^+}} > 1$. By (H_1) and Lemma (5), we get

$$\begin{aligned}
J(u) &= \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{a(x)}{q(x)}|\nabla u|^{q(x)}\right) dx + \int_\Omega \left(\frac{1}{p(x)}|u|^{p(x)} + \frac{a(x)}{q(x)}|u|^{q(x)}\right) dx - \int_{\partial\Omega} G(x,u)\, d\sigma \\
&\geq \frac{1}{q^+}\hat{\rho}_{\mathcal{H}}(u) - c_3\int_{\partial\Omega} |u|\, d\sigma - c_4\int_{\partial\Omega} |u|^{\alpha(x)}\, d\sigma \\
&\geq \frac{1}{q^+}\|u\|^{p^-} - c_5\|u\| - c_6|u|^{\alpha(\xi)}_{\alpha(x),\partial\Omega} \quad \text{where} \ \ \xi \in \partial\Omega \\
&\geq \begin{cases} \frac{1}{q^+}\|u\|^{p^-} - c_5\|u\| - c_6, & \text{if} \ \ |u|_{\alpha(x),\partial\Omega} \leq 1 \\ \frac{1}{q^+}\|u\|^{p^-} - c_5\|u\| - c_6\left(\alpha_k\|u\|\right)^{\alpha^+}, & \text{if} \ \ |u|_{\alpha(x)} > 1 \end{cases} \\
&\geq \frac{1}{q^+}\|u\|^{p^-} - c_5\|u\| - c_6\left(d_k\|u\|\right)^{\alpha^+} - c_6,
\end{aligned}$$

where c_i, $i = 3, 4, 5, 6$ are positive constants.
Because $q^+ < \alpha^-$ and $d_k \to 0$ as $k \to +\infty$, then,

$$J(u) \geq (\frac{1}{q^+} - \frac{1}{\alpha^-})R_k^{p^-} - c_5 R_k - c_6 \to +\infty \quad \text{as} \quad k \to +\infty,$$

which implies (A_1).

Verification of $(\mathbf{A_2})$. Because $Y_k = \bigoplus_{j=1}^k E_j$ is finite-dimensional space, all norms are equivalent. Then there exists $N_k > 0$, for all $u \in Y_k$ with $\|u\|$ is large enough, we obtain

$$I(u) \leq \frac{1}{p^-}\hat{\rho}_{\mathcal{H}}(u) \leq \frac{1}{p^-}\|u\|^{q^+} \leq N_k|u|^{q^+}_{q^+}. \tag{22}$$

Moreover, it follows from (H_2) that there exist $H_k > 0$ such that for every $|t| \geq H_k$, we get

$$G(x,t) \geq 2N_k|t|^{q^+} \quad \text{for a.e. } x \in \partial\Omega.$$

Therefore, for every $(x,t) \in \partial\Omega \times \mathbb{R}$, we obtain

$$G(x,t) \geq 2N_k|t|^{q^+} - G_k,$$

where $G_k = \max\limits_{|t|\leq H_k} G(x,t)$.
Combining this with (22), for $u \in Y_k$ such that $\|u\| = \gamma_k > \eta_k$, we find

$$\begin{aligned} J(u) &= I(u) - \int_{\partial\Omega} G(x,u)\, d\sigma \\ &\leq N_k|u|_{q^+}^{q^+} - 2N_k|u|_{q^+}^{q^+} + G_k|\partial\Omega| \\ &\leq -N_k|u|_{q^+}^{q^+} + G_k|\partial\Omega| \\ &\leq -\frac{1}{p^-}\|u\|^{q^+} + G_k|\partial\Omega|. \end{aligned}$$

Consequently, from the above inequalities, for γ_k large enough $(\gamma_k > \eta_k)$, we obtain

$$c_k := \max\{J(u) : u \in Y_k, \|u\| = \gamma_k\} \leq 0,$$

which implies (A_2).

Therefore, the proof is completed by applying the Fountain Theorem.

5 Acknowledgement

The authors would like to thank the referees for their valuable comments and suggestions, which have improved the quality of this paper.

References

Ayoujil, A. (2014). On the superlinear Steklov problem involving the $p(x)$-Laplacian. *Electronic Journal of Qualitative Theory of Differential Equations*, *2014*(38), 1-13.

Bartolo, P., Benci, V. & Fortunato, D. (1983). Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity. *Nonlinear Analysis: Theory, Methods & Applications*, *7*(9), 981-1012.

Benkirane, A., Sidi El Vally, M. (2014). Variational inequalities in Musielak–Orlicz–Sobolev spaces. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, *21*(5), 787-811.

Berdawood, K.A., Nachaoui, A., Saeed, R., Nachaoui, M., & Aboud, F. (2020). An alternating procedure with dynamic relaxation for Cauchy problems governed by the modified Helmholtz equation. *Advanced Mathematical Models & Applications, 5*(1), 131-139.

Cerami, G. (1978). An existence criterion for the critical points on unbounded manifolds. *Istit. Lombardo Accad. Sci. Lett. Rend. A*, *112*(2), 332-336.

Colasuonno, F., Squassina, M. (2016). Eigenvalues for double phase variational integrals. *Annali di Matematica Pura ed Applicata*, *195*(6), 1917–1959.

Crespo-Blanco, A., Gasinski, L., Harjulehto, P. & Winkert, P. (2021). A new class of double phase variable exponent problems: Existence and uniqueness. arXiv preprint, *arXiv: 2103.08928*.

Cui, N., Sun, H-R. (2021). Existence and multiplicity results for double phase problem with nonlinear boundary condition. *Nonlinear Analysis: Real World Applications*, 60, 103307.

Fan, X. (2012). An imbedding theorem for Musielak-Sobolev spaces. *Nonlinear Analysis*, *75*(4), 1959-1971.

Fan, X.L., Zhang, Q.H. (2003). Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Analysis*, *52*(8), 1843–1852.

Harjulehto, P., Hasto, P. (2019). Orlicz Spaces and Generalized Orlicz Spaces. *Lecture Notes in Mathematics*, 2236. Springer, Cham.

Jeanjean, L. (1999). On the existence of bounded Palais-Smale sequences and application to Landesman-lazer type problem set on $\mathbb{R}^N$. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, *129*(4), 787–809.

Liu, L., Dai, G. (2018). Existence and multiplicity results for double phase problem. *Journal of Differential Equations*, *265*(9), 4311–4334.

Musielak, J. (1983). Orlicz Spaces and Modular Spaces. *Lecture Notes in Math*, vol. 1034, Springer, Berlin.

Nachaoui, A., Nachaoui, M. & Gasimov. B. (2021). Parallel numerical computation of an analytical method for solving an inverse problem. *Advanced Mathematical Models & Applications, 6(2)*, 162-173.

Rasheed, S.M., Nachaoui, A., Hama, M.F. & Jabbar, A.K. (2021). Regularized and preconditioned conjugate gradient like-methods methods for polynomial approximation of an inverse Cauchy problem. *Advanced Mathematical Models & Applications, 6*(2), 89-105.

Yang, J., Chen, H. & Liu, S. (2020). Existence and Multiplicity of Solutions for a Class of Anisotropic Double Phase Problems. *Advances in Mathematical Physics*, 2020.

Zang, A. (2008). $p(x)$-Laplacian equations satisfying Cerami condition. *J. Math. Anal. Appl.*, *337*(1), 547–555.

Zhao, J.F. (1991). Structure theory of Banach spaces. *Wuhan University Press*, Wuhan (in Chinese).

Zhikov, V.V. (1986). Averaging of functionals of the calculus of variations and elasticity theory. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, *50*(4), 675–710.

Zhikov, V.V. (1995). On Lavrentiev's phenomenon. *Russian Journal of Mathematical Physics*, *3*(0), 2.

Zhikov, V.V., Kozlov, S.M. & Oleinik, O.A. (1994). *Homogenization of Differential Operators and Integral Functionals*. Berlin: Springer-Verlag.

Zou, W. (2001). Variant fountain theorems and their applications. *Manuscripta Mathematica*, *104*(3), 343-358.